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QCD Matching Conditions at Thresholds

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Abstract

The use of MS-like renormalization schemes in QCD requires an implementation of nontrivial matching conditions across thresholds, a fact often overlooked in the literature. We shortly review the use of these matching conditions in QCD and check explicitly that the prediction for $\alpha_s(M_Z)$, obtained by running the strong coupling constant from the M_τ scale, does not substantially depend on the exact value of the matching point chosen in crossing the b-quark threshold when the appropriate matching conditions are taken into account.

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During the last years a great effort has been done at LEP in order to measure the strong coupling constant $\alpha_s(M_Z)$ at the Z mass scale [1, 2, 3, 4]. This measurement has been of crucial importance since it allowed, within the experimental errors, the running of the strong coupling constant to be checked from low energies to the electroweak scale. However, by going to higher orders in the renormalization group equations, some confusion has arisen in the literature on the different prescriptions one could use to cross thresholds in the evolution of the running coupling constant. The problem appears when working in MS-like renormalization schemes: since these are mass-independent, the decoupling theorem of Appelquist-Carazzone [5] is not fulfilled in "non-physical" quantities such as beta functions or coupling constants. Only in physical quantities particles with large masses do decouple. Logarithms of large masses induced by the renormalization group equations in the couplings are cancelled against other logarithms that appear in the calculation of physical observables. This is obviously an inconvenient, since a lot of effort must be invested in intermediate stages of a calculation to compute terms that will cancel in physical quantities. To remedy this problem the standard procedure has been the use of the effective field theory language [6, 7, 8]. For example, in QCD with a heavy quark and $N-1$ light quarks, one builds a theory with N quarks and an effective field theory with $N-1$ quarks. Around the threshold of the heavy quark one requires agreement of the two theories. This gives a set of matching equations that relate the couplings of the theory with N quarks with the couplings of the theory with $N-1$ quarks. This way, below the heavy quark threshold one can work with the effective theory, but using effective couplings. Then, by construction, decoupling is trivial. This procedure is equivalent to other renormalization schemes and allows us to correctly obtain the asymptotic value of the coupling constant. The price one has to pay is that coupling constants might not be continuous at thresholds. All this machinery is well established since the early 80's [6, 7, 8, 9, 10] and matching conditions were computed at the one-loop level [7, 8] and at the two-loop level [9, 10] for general gauge theories. It also seems to be well known for people working with $GUTs$ [8] where, in general, special attention has been paid to matching conditions at the different thresholds. However, the fact that one has to use appropriate matching conditions in passing thresholds has been frequently overlooked in the running of the QCD coupling constant by just taking a continuous coupling constant across thresholds. Then, the final results depend strongly on the exact scale one uses to connect the couplings [11, 12, 13, 4]. To solve this ambiguity some of the authors [11] vary the matching scale between 0.75 and 2.5 times the mass of the heavy quark; others [12, 13] use directly $\mu_{th} = 2m_q$, and yet others [4] determine $\alpha_s(M_Z)$ with both $\mu_{th} = m_q$ and $\mu_{th} = 2m_q$ and take the average. Here we will show that when appropriate matching conditions are taken into account the final answer does not depend on the exact μ_{th} used to connect the couplings.

Although most of the points discussed in this paper are well known in some circles, given the confusion that exists in the literature and the importance of the subject we found it convenient to recall what the correct matching conditions are and to show that when they are consistently taken into account the dependence on the renormalization scale cancels (at least at the order the calculation is done). Consistency requires that if the evolution of the gauge coupling constant is done at n loops, matching conditions should be imposed using $n-1$ loop formulae [7]; then the residual dependence on the renormalization scale is of order $n+1$. We will show this, explicitly, when running the QCD gauge constant from the τ mass to the Z mass passing through the b threshold.

The renormalization group equations in QCD for the strong gauge coupling constant and the quark masses are

$$\frac{d\alpha_s}{dt} = -\alpha_s^2 \left(\beta_0 + \frac{\beta_1}{4\pi}\alpha_s + \frac{\beta_2}{(4\pi)^2}\alpha_s^2 + \cdots \right) \quad (1)$$

$$\frac{dm^2}{dt} = -4\pi \left(\gamma_0 \frac{\alpha_s}{\pi} + \gamma_1 \left(\frac{\alpha_s}{\pi}\right)^2 + \cdots \right) m^2 \quad (2)$$

where

$$t = \frac{1}{4\pi} \log\left(\frac{\mu^2}{\mu_0^2}\right) \quad (3)$$

and μ_0 is some reference point.

The β coefficients governing the evolution of the gauge coupling constant are

$$\begin{aligned} \beta_0 &= 11 - \frac{2}{3}N_F \\ \beta_1 &= 102 - \frac{38}{3}N_F \\ \beta_2 &= \frac{1}{2}\left(2857 - \frac{5033}{9}N_F + \frac{325}{27}N_F^2\right) \end{aligned} \quad (4)$$

with N_F the number of quark flavours with mass lower than the renormalization scale μ. The first two coefficients are scheme-independent (in MS-like schemes) but the higher-order coefficients depend on the renormalization conditions [14]. We give β_2 in the $\overline{MS}$ scheme.

The quark mass anomalous dimensions are

$$\begin{aligned} \gamma_0 &= 2 \\ \gamma_1 &= \frac{101}{12} - \frac{5}{18}N_F \ . \end{aligned} \quad (5)$$

Integration of eq. (1) can be performed by first inverting the series on the right-hand side of eq. (1) and then integrating on α and t. Finally, one can solve for α, at the required order, by using iterative methods. The result we obtain can be written in the following form

$$\alpha_s(\mu) = \alpha_s^{(1)}(\mu) + \alpha_s^{(2)}(\mu) + \alpha_s^{(3)}(\mu) + \cdots \quad (6)$$

where $\alpha_s^{(1)}$, $\alpha_s^{(2)}$, $\alpha_s^{(3)}$ represent the one-, two- and three-loop contributions respectively, and are given by

$$\alpha_s^{(1)}(\mu) = \frac{\alpha_s(\mu_0)}{1 + \alpha_s(\mu_0)\beta_0 t} \quad (7)$$

$$\alpha_s^{(2)}(\mu) = -(\alpha_s^{(1)})^2 b_1 \log K(\mu) \quad (8)$$

$$\alpha_s^{(3)}(\mu) = (\alpha_s^{(1)})^3 \left(b_1^2 \log K(\mu) \left(\log K(\mu) - 1\right) - \left(b_1^2 - b_2\right)\left(1 - K(\mu)\right)\right) \ , \quad (9)$$

where

$$b_1 = \frac{\beta_1}{4\pi\beta_0}, \qquad b_2 = \frac{\beta_2}{(4\pi)^2\beta_0}, \qquad K(\mu) = \frac{\alpha_s(\mu_0)}{\alpha_s^{(1)}(\mu)} \quad (10)$$

The running quark mass can also be obtained analytically at the one-loop level, from eq. (2). The result is

$$m^2(\mu) = m^2(\mu_0)(1 + \alpha_s(\mu_0)\beta_0 t)^{-4\gamma_0/\beta_0} \ . \quad (11)$$

For $\alpha_s(\mu_0)\beta_0 t << 1$ the result is independent of β_0 and can be written as (we use $\gamma_0 = 2$)

$$m^2(\mu) = m^2(\mu_0)(1 - 8\alpha_s(\mu_0)t + \cdots) \ . \tag{12}$$

This expression can be used as long as μ is not very different from μ_0; in particular we could use it to simplify the matching conditions.

Conventionally [15, 14], higher-order RGEs are solved by doing a power series expansion in $1/L$ with $L = \log(\mu^2/\Lambda^2)$. This solution is given in terms of the QCD scale Λ, which is defined in such a way that it is renormalization-group-invariant but scheme-dependent and the so-called invariant mass $\hat{m}$. Passing of thresholds is implemented by requiring continuity of the couplings at threshold, which in turn requires defining different Λ's for different N_F. For our purposes we prefer to use the solutions given above because they allow us to work more easily with scale-dependent matching conditions.

In the MS scheme, or any of its simple modifications such as $\overline{MS}$, the beta function governing the running of the strong coupling constant is independent of quark masses. Then, contrary to what happens in momentum-subtraction schemes (MO), the Appelquist-Carazzone theorem [5] that states, when it can be applied, that the heavy particles decouple at each order of perturbation theory is not realized in a trivial way. The decoupling of the heavy particles is fulfilled in physical quantities, but coupling constants and beta functions do not exhibit it.

To obtain decoupling in MS schemes we need to build in the decoupling region, $\mu \ll M$, an effective field theory that behaves as if only the light degrees of freedom were present. Matching conditions connect the parameters of the low-energy effective Lagrangian with the parameters of the full theory. This can be done by evaluating some Green functions in perturbation theory with both the full and the effective theories, then require they are the same, up to terms $O(1/M)$, for values of the renormalization scale just around the threshold. Then, the coupling constant of the effective theory can be expressed as a power series expansion in the coupling of the full theory with coefficients that depend on $\log(M/\mu)$. In order to obtain a good approximation using only the first few terms in the perturbative expansion, we have to evaluate matching conditions in a region where $M/\mu \sim O(1)$. However, the results of these calculations should not depend on exactly which μ is chosen.

One-loop matching conditions have been obtained in [7, 8] for a general gauge theory. To obtain matching conditions in QCD at the two-loop level several approaches have been pursued. Ovrut and Schnitzer [9] computed the gluon self energies at the two-loop level with both the full and the effective theories and then required matching in the threshold region. We will follow a more direct approach, devised by Bernreuther and Wetzel [10]. Using the MO scheme as an intermediate stage, these authors were able to relate the $\overline{MS}$ coupling constant $\alpha_{\overline{MS}}(\mu)$, with N_F quark flavours, with the gauge coupling constant $\alpha_{\overline{MS}}^{-}(\mu)$ of the effective field theory with $N_F - 1$ quark flavours in which a heavy quark with $m_{\overline{MS}}$ mass has been integrated out. This is because in momentum subtraction schemes the decoupling theorem is also realized in the coupling constants. The obtained relation has the following form:

$$\alpha_{\overline{MS}}^{-} = \alpha_{\overline{MS}} \left(1 + \sum_{k=1}^{\infty} \alpha_{\overline{MS}}^k C_k(x)\right) \tag{13}$$

with

$$x = \frac{1}{4\pi} \log(m_{\overline{MS}}^2/\mu^2) \tag{14}$$

In order to calculate the coefficients C_k Bernreuther and Wetzel impose the RGEs, eq. (1) and eq. (2), on $\alpha_{\overline{MS}}$, $\alpha_{\overline{MS}}^{-}$ and $m_{\overline{MS}}$, and obtain for the first two coefficients a set of coupled first-order linear differential equations depending only on the beta and gamma functions of the full and the effective theories. By solving them they found for a general $SU(N)$ group the following result valid for the $\overline{MS}$ scheme[1]

$$\begin{aligned} C_1 &= \frac{2}{3}\left(x + \frac{1}{8\pi}\frac{\partial}{\partial D}\,\mathrm{T}r\{I\}|_{D=4}\right) \\ C_2 &= [C_1(x)]^2 + \frac{1}{2\pi}\left(\frac{5}{3}C_2(G) - C_2(R)\right)x + \frac{1}{9\pi^2}C_2(G) \\ &\quad -\frac{17}{96\pi^2}C_2(R) + \frac{1}{32\pi^2}\left(\frac{5}{3}C_2(G) - C_2(R)\right)\frac{\partial}{\partial D}\,\mathrm{T}r\{I\}|_{D=4} \end{aligned} \tag{15}$$

with $C_2(G) = N$ and $C_2(R) = \dfrac{N^2-1}{2N}$ the Casimir operator eigenvalues of the adjoint and fundamental representations, respectively, and D the space-time dimension. A technical point about the trace of the identity in the Dirac space, $\mathrm{T}r\{I\}$, should be discussed here. Strictly, in a general D-dimensional space, D even, the only irreducible representation of

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{16}$$

has dimension $f(D) = 2^{D/2}$. However, we can choose $\mathrm{T}r\{I\} = f(D) = 4$, or any other smooth function with $f(4) = 4$. Different choices of $f(D)$ lead to different trivial modifications of the $\overline{MS}$ renormalization scheme. However, as can be seen in eq. (15), different choices give quite different matching conditions. Hence, in order to specify completely the renormalization scheme within the MS-like schemes one should also specify which convention has been used for $\mathrm{T}r\{I\}$. Here we will use the usual convention among phenomenology papers, i.e. $\mathrm{T}r\{I\} = 4$. Then for QCD we have the following two-loop matching condition to connect the theory with $N-1$ quarks with the theory with N quarks at the q-quark threshold [10]

$$\begin{aligned} \alpha_{N-1}(\mu_{th}) &= \alpha_N(\mu_{th}) + \frac{\alpha_N^2(\mu_{th})}{3\pi}\log\frac{m_q(\mu_{th})}{\mu_{th}} \\ &+ \frac{\alpha_N^3(\mu_{th})}{9\pi^2}\left(\left(\log\frac{m_q(\mu_{th})}{\mu_{th}}\right)^2 + \frac{33}{4}\log\frac{m_q(\mu_{th})}{\mu_{th}} + \frac{7}{8}\right). \end{aligned} \tag{17}$$

Here, μ_{th} is the value at which we require matching. As commented, this equation is valid for arbitrary values of μ_{th} as long as it is not far away from $m_q(m_q)$. Should we use instead $\mathrm{T}r\{I\} = 2^{D/2}$, the logarithm in the second term would be changed to $\log\sqrt{2}m_q/\mu_{th}$ changing completely the behaviour of the matching conditions. For example, it is clear from the above equation that one can always choose a μ_{th} in order to make the coupling continuous across thresholds. Using only the one-loop matching condition, i.e. only the first two terms in the right-hand side of eq. (17), and with $\mathrm{T}r\{I\} = 4$, we should require $\alpha_{N-1}(m_q) = \alpha_N(m_q)$. Using the two-loop matching condition, the matching point is slightly different [2]. However, if a scheme with $\mathrm{T}r\{I\} = 2^{D/2}$ is used the matching point is

[1]The solution of the two (for two loops) differential equations depends on two scheme-dependent arbitrary constants. To fix them one has to perform a complete calculation in the scheme one is interested in.

[2]One can still impose $\alpha_{N-1}(m_q) = \alpha_N(m_q)$ at the two-loop level, as Marciano does [14], but this requires a slight modification of the $\overline{MS}$ scheme in order to absorb the non-logarithmic term in eq. (17).

found around $\sqrt{2}m_q$. In what follows we will keep the couplings discontinuous and check the invariance of the final result with respect to the chosen matching scale μ_{th}.

Equation (17) can be simplified by using eq. (12) to remove the dependence on the running mass and leave the result in terms of $m_q \equiv m_q(m_q)$, the $\overline{MS}$ running mass evaluated at its own value. Since we are running from low energies to high energies it is also better to use the inverted equation. Thus, our matching condition at the threshold of the quark q will be:

$$\alpha_N(\mu_{th}) = \alpha_{N-1}(\mu_{th}) - \frac{\alpha^2_{N-1}(\mu_{th})}{3\pi}\log\frac{m_q}{\mu_{th}}$$
$$+\frac{\alpha^3_{N-1}(\mu_{th})}{9\pi^2}\left(\left(\log\frac{m_q}{\mu_{th}}\right)^2 - \frac{57}{4}\log\frac{m_q}{\mu_{th}} - \frac{7}{8}\right) \ . \tag{18}$$

We start from a scale below the bottom-quark threshold, where we know the value of the strong coupling constant[3] , e.g. M_τ

$$M_\tau = 1776.9 \pm 0.7 \text{ MeV}$$
$$\alpha_3(M_\tau) = 0.36 \pm 0.03 \ , \tag{19}$$

We use[4] eq. (18) with $\mu_{th} = M_\tau$ and $m_q = m_c$ to obtain $\alpha_4(M_\tau)$ in terms of $\alpha_3(M_\tau)$. Then we evolve $\alpha_4(\mu)$ until the Z boson mass scale by imposing matching conditions at an arbitrary intermediate scale μ_{th} around m_b. To run $\alpha_4(\mu)$ from M_τ until μ_{th} we use eq. (6) with $\mu_0 = M_\tau$ and four-quark beta functions. Then at μ_{th} we impose the matching condition eq. (18) with $m_q = m_b$ to obtain $\alpha_5(\mu_{th})$ in terms of $\alpha_4(\mu_{th})$. Finally, to run $\alpha_5(\mu)$ from μ_{th} to M_Z we use again eq. (6), but now with $\mu_0 = \mu_{th}$, and with five-quark beta functions. The evolution is consistent, i.e. to the same order, if n-loop beta functions are used together with matching conditions evaluated at the $(n-1)$-loop level. Firstly, we run $\alpha_s(\mu)$ at the one-loop order, eq. (7), with matching conditions at tree level, i.e. taking $\alpha_4(\mu_{th}) = \alpha_5(\mu_{th})$ with μ_{th} around m_b. After that, we calculate $\alpha_s(M_Z)$ by running $\alpha_s(\mu)$ with two-loop beta functions and imposing matching conditions at the one-loop order, eq. (18), but taking only the first two terms on its right-hand side. And finally, we evaluate $\alpha_s(M_Z)$ according to the three-loop evolution, eq. (9), with matching conditions at two-loop level, eq. (18).

We show the final results in fig. 1. We can clearly see that, as expected, the variation of the final prediction on $\alpha_5(M_Z)$, as we vary the matching point around the bottom quark mass, is of the same order of magnitude as the next-order corrections; for three-loop beta functions and two-loop matching conditions it is practically flat. For comparison purposes we also give the error bar induced from the error in $\alpha_s(M_\tau)$. Given the level of accuracy, two-loop beta functions and one-loop matching conditions seem to be good enough for all purposes.

In the preceding section we directly used the matching equation to evaluate $\alpha_4(M_\tau)$ in terms of $\alpha_3(M_\tau)$. We could proceed in that way because the mass of the c-quark and the mass of the τ are not so different; the logarithms in the matching equation are therefore

[3]The value of $\alpha_3(M_\tau)$ has been extracted at the two-loop level from hadronic τ decays in [16]. We took their result directly.

[4] Our starting point for the quark masses are the so-called Euclidean masses [17], $M_b^E = 4.23 \pm 0.05$ GeV and $M_c^E = 1.26 \pm 0.02$ GeV, from which we extract the $\overline{MS}$ masses $m_b \equiv m_b(m_b) = 4.3 \pm 0.2$ GeV and $m_c \equiv m_c(m_c) = 1.3 \pm 0.2$ GeV.

not large. Alternatively one could try to run $\alpha_3(M_\tau)$ until some intermediate scale μ_{th} around the charm threshold. Then, impose eq. (18) with $m_q = m_c$ to get $\alpha_4(\mu_{th})$ and run it until the bottom-quark threshold. This time, since we are interested only in the error induced by crossing the charm threshold we will use eq. (18) with $m_q = m_b$ and $\mu_{th} = m_b$ fixed to obtain $\alpha_5(m_b)$. Finally we run $\alpha_5(\mu)$ from m_b until M_Z. Of course this procedure should give, within the level of precision of the order considered, the same result as before. In fig. 2 we give $\alpha_5(M_Z)$ as a function of the matching point μ_{th} taken for the charm threshold. Although now the result depends on the matching scale μ_{th}, this dependence is always a next-order correction as long as the matching conditions are implemented correctly. Clearly this procedure is potentially very dangerous since an incorrect use of matching conditions could lead to a false strong dependence on the matching scale. A similar consideration could be applied to the bottom quark threshold. Then, probably the safest procedure to run $\alpha_3(M_\tau)$ until the Z mass would be to use first eq. (18) with $m_q = m_c$ and $\mu_{th} = M_\tau$ to get $\alpha_4(M_\tau)$ in terms of $\alpha_3(M_\tau)$, then use again eq. (18) with $m_q = m_b$ and $\mu_{th} = M_\tau$ to get $\alpha_5(M_\tau)$ in terms of $\alpha_4(M_\tau)$. Finally we should run $\alpha_5(\mu)$ from M_τ until M_Z with the full five-quark renormalization group. This procedure is justified since the masses of the b-quark, c-quark and τ-lepton are not so different as to spoil the validity of the matching equation. Working in this way we arrived at the value[5] $\alpha_5(M_Z) = 0.123 \pm 0.004$, in complete agreement with our previous result.

To conclude, we would like to remark on the following points:

- Only in MO-like schemes, where Appelquist-Carazzone is realized in both beta functions and coupling constants, the strong coupling constant $\alpha_s(\mu)$ is continuous. In $\overline{MS}$-like schemes one should build a low-energy effective field theory and write scale-dependent matching conditions in order to connect the parameters of the theories on both sides of the threshold μ_{th}. Then, for general values of μ_{th} the couplings are not continuous although in the case of only one coupling constant it is always possible to find a particular μ_{th} that makes the coupling continuous.

- Evolution is consistent, i.e. to the same order, if the evolution of the gauge coupling constant at the n-loop order is accompanied by matching conditions at the $(n-1)$-loop level.

- Different choices for the trace in Dirac space, i.e. $\mathrm{T}r\{I\} = 4$ or $\mathrm{T}r\{I\} = 2^{D/2}$, give rise to different trivial modifications of the $\overline{MS}$ scheme with quite different matching conditions. Should one insist on having a continuous coupling across thresholds, it is clear from the discussion that the precise matching point will depend on the choice for the trace in Dirac space. For instance, working with two-loop beta functions one should take

$$\begin{aligned} &\mu_{th} = m_b \quad \text{if} \quad \mathrm{T}r\{I\} = 4 \\ &\mu_{th} = \sqrt{2} m_b \quad \text{if} \quad \mathrm{T}r\{I\} = 2^{D/2} \quad . \end{aligned}$$

By running the strong coupling constant from the M_τ scale to the M_Z scale, we have checked explicitly that the final answer is not sensitive to the exact value of the matching point, μ_{th}, used in crossing the m_b threshold as long as the right matching conditions are consistently taken into account (fig. 1). Similar considerations apply when crossing the m_c threshold (fig. 2). Finally we have shown that the correct result can be obtained by

[5]We include the error induced by the errors in the quark masses, which is about 0.001 in $\alpha_5(M_Z)$.

using the matching conditions to find $\alpha_5(M_\tau)$ in terms of $\alpha_3(M_\tau)$ and then run it with the full five-quark renormalization group until the M_Z scale.

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References

[1] L3 Collaboration, Phys. Lett. **B284** (1992) 471

[2] ALEPH Collaboration, Phys. Lett. **B284** (1992) 163

[3] DELPHI Collaboration, Z. Phys. **C54** (1992) 55

[4] OPAL Collaboration, Z. Phys. **C55** (1992) 1

[5] T. Appelquist and J. Carazzone, Phys. Rev. **D11** (1975) 2856

[6] E. Witten, Nucl. Phys. **B104** (1976) 445

[7] S. Weinberg, Phys. Lett. **91B** (1980) 51

[8] L. Hall, Nucl. Phys. **B178** (1981) 75

[9] B. Ovrut and H. Schnitzer, Nucl. Phys. **B179** (1981) 381; **189** (1981) 509

[10] W. Bernreuter and W. Wetzel, Nucl. Phys. **B197** (1982) 228; W. Wetzel, Nucl. Phys. **B196** (1982) 259; W. Bernreuter, Ann. Phys. **151** (1983) 127

[11] G. Altarelli, "QCD and experiment: Status of α_s", CERN-TH.623/92 (1992)

[12] A. Petermann, "$\alpha_s(Q^2)$ extrapolation with analytic thresholds", CERN-TH.6487/92 (1992)

[13] S. Bethke and S. Catani, "A summary of α_s measurements", CERN-TH.6484/92 (1992)

[14] W.J. Marciano, Phys. Rev. **D29** (1984) 580

[15] S. Narison, Phys. Rep. **84** (1982) 263

[16] S. Pich, "QCD predictions for the tau hadronic width and determination of $\alpha_s(M_Z)$", CERN.TH-6738/92 (1992)

[17] S. Narison, Phys. Lett. **B197** (1987) 405

Figure captions

Figure 1: Strong coupling constant at the M_Z scale, obtained by running the coupling from its value at the M_τ scale ($\alpha_3(M_\tau) = 0.36 \pm 0.03$), as a function of the matching point taken to cross the b-quark threshold. The long-dashed line is obtained by using one-loop beta functions and tree-level matching conditions. The dashed line is obtained with two-loop beta functions and one-loop matching conditions, and the solid line is obtained with three-loop beta functions and two-loop matching conditions. Error bars on the final three-loop result are given for comparison purposes with other $\alpha_s(M_Z)$ results.

Figure 2: Same as in fig. 1, but varying the matching point around the c-quark threshold. The matching point for the b-quark is now fixed at m_b.